Geometric Series

$$S_N = \sum_{k=0}^{N} a \cdot r^k = N^{\text{th}} \text{ partial sum}$$
(finite geometric series)

$$\lim_{N\to\infty} S_N = S_\infty \leftarrow \text{sometimes exists}$$
$$\sum_{k=0}^{\infty} a \cdot r^k .$$

Formula for S_N:

$$S_N = \sum_{k=0}^{N} a \cdot r^k = a + \cancel{ar} + \cancel{ar^2} + \cancel{ar^3} + \cdots + \cancel{ar^{N-1}} + \cancel{ar^N}$$

$$\Rightarrow rS_N = \cancel{ar} + \cancel{ar^2} + \cancel{ar^3} + \cdots + \cancel{ar^N} + ar^{N+1}$$

$$\Rightarrow S_N - rS_N$$
$$(1-r)S_N = a - ar^{N+1}$$

$$\Rightarrow \boxed{\boxed{S_N = \frac{a(1-r^{N+1})}{1-r}} = \sum_{k=0}^{N} ar^k} \quad \boxed{\text{if } r \neq 1}$$

$\left(S_N = (N+1)a \text{ if } r=1 \right)$

formula for the sum of a finite geometric series.

$$\lim_{N\to\infty} r^{N+1} = \begin{cases} 0 & \text{if } |r|<1 \\ \text{DNE} & \text{if } r>1 \text{ or } r\le -1 \quad (-1)^{N+1} \text{ diverges.} \\ 1 & \text{if } r=1. \end{cases}$$

If $|r|<1$, the geometric series converges, and

$$\sum_{k=0}^{\infty} ar^k = S_\infty = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{a(1-r^{N+1})}{1-r}$$

$$= \frac{a\left(1-\lim_{N\to\infty} r^{N+1}\right)}{1-r} = \frac{a(1-0)}{1-r} = \frac{a}{1-r}$$

$$\boxed{\therefore \text{ If } |r|<1, \ \sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}}$$

sum of ∞ geometric series.

$$\boxed{\text{If } |r|\geq 1, \ \sum_{k=0}^{\infty} ar^k \text{ diverges.}}$$

$\boxed{\text{Ex}}$ Compute

$$48 - 7 + \frac{7}{3} - \frac{7}{9} + \frac{7}{27} - \frac{7}{81} + \frac{7}{243} - \cdots$$

This is $48 + (-7) + (-7)\left(-\frac{1}{3}\right) + (-7)\left(\frac{-1}{3}\right)^2 + (-7)\left(\frac{-1}{3}\right)^3 + \cdots$

$$= 48 + \underbrace{\sum_{k=0}^{\infty} (-7)\left(-\frac{1}{3}\right)^k}_{\text{geometric}}$$

a = first term $= -7$

$r = \frac{-1}{3}$

$\frac{a}{1-r}$

$$= 48 + \frac{a}{1-r} = 48 + \frac{-7}{1-(-\frac{1}{3})}$$

$$= 48 + \frac{-7}{4/3} = 48 + \frac{-21}{4}$$

$$= \frac{192 - 21}{4} = \boxed{\frac{171}{4}}.$$

Example (Annuity). Ava is depositing \$200 per month into a bank account that earns 4% annual interest, compounded monthly. How much will Ava have when she retires in 30 years?

---

month 1 total $= \underbrace{\$200}_{\text{1st deposit.}}$

month 2 total $= \$200 + \underbrace{\$200\left(\frac{.04}{12}\right)}_{\text{interest}} + \underbrace{\$200}_{\text{2nd deposit}}$

$$= \$200\left(1+\frac{.04}{12}\right) + \$200$$

month 3 total $= \left(\$200\left(1+\frac{.04}{12}\right) + \$200\right) + \underbrace{\left(200\left(1+\frac{.04}{12}\right)+200\right)\left(\frac{.04}{12}\right)}_{\text{interest}}$

$$+ \underbrace{\$200}_{\text{3rd deposit}}$$

$$= \left(\$200\left(1+\frac{.04}{12}\right)+\$200\right)\left(1+\frac{.04}{12}\right) + \$200$$

$$= \$200\left(1+\frac{.04}{12}\right)^2 + \$200\left(1+\frac{.04}{12}\right) + \$200$$

total $= \$200 + 200\left(1+\frac{.04}{12}\right) + 200\left(1+\frac{.04}{12}\right)^2$

month 4 total $= \$200 + 200\left(1+\frac{.04}{12}\right) + 200\left(1+\frac{.04}{12}\right)^2 + 200\left(1+\frac{.04}{12}\right)^3$

$\downarrow$

month 360

$$\text{total} = \$200 + \$200\left(1+\tfrac{.04}{12}\right) + \$200\left(1+\tfrac{.04}{12}\right)^2 + \dots$$

$$\dots + \$200\left(1+\tfrac{.04}{12}\right)^{359}$$

finite geometric

$$a = 200$$

$$r = 1 + \tfrac{.04}{12}$$

$$N = 359$$

$$\text{total} = \frac{a(1-r^{N+1})}{1-r} = \frac{200\left(1-\left(1+\frac{.04}{12}\right)^{360}\right)}{1-\left(1+\frac{.04}{12}\right)}$$

$$= \frac{200\left(1-\left(1+\frac{.04}{12}\right)^{360}\right)}{-\frac{.04}{12}}$$

$$= \frac{200 \cdot 12}{-.04}\left(1-\left(1+\frac{.04}{12}\right)^{360}\right)$$

$$\approx \boxed{\$138{,}810.}$$

How much was payments? How much was interest earned?

$360 \cdot 200 = \$72000$ payments

$(\text{interest } \$) = \$138{,}810 - \$72{,}000$

$= \$66{,}810.$

Another kind of series where we can calculate the infinite sum:

Ex Find

$$S = \frac{1}{4}\cdot\frac{1}{7} + \frac{1}{5}\cdot\frac{1}{8} + \frac{1}{6}\cdot\frac{1}{9} + \cdots$$

Step 1: Write in summation format.

$$S = \sum_{k=4}^{\infty} \frac{1}{k(k+3)}$$

Step 2: Partial fractions

$$\frac{1}{k(k+3)} = \frac{A}{k+3} + \frac{B}{k}$$

$$1 = Ak + B(k+3)$$

$$k=0 \quad 1 = B(3) \Rightarrow B = \frac{1}{3}$$

coefficient of k: $0 = A + B \Rightarrow A = -\frac{1}{3}$.

$$\Rightarrow S = \sum_{k=4}^{\infty}\left(\frac{-\frac{1}{3}}{k+3} + \frac{\frac{1}{3}}{k}\right)$$

Step 3. Write out sum again.

$$S = \left(\cancel{\frac{-\frac{1}{3}}{7}} + \frac{\frac{1}{3}}{4}\right) + \left(\cancel{\frac{-\frac{1}{3}}{8}} + \frac{\frac{1}{3}}{5}\right) + \left(\cancel{\frac{-\frac{1}{3}}{9}} + \frac{\frac{1}{3}}{6}\right)$$

$$+ \left(\cancel{\frac{-\frac{1}{3}}{10}} + \cancel{\frac{\frac{1}{3}}{7}}\right) + \left(\cancel{\frac{-\frac{1}{3}}{11}} + \cancel{\frac{\frac{1}{3}}{8}}\right) + \left(\cancel{\frac{-\frac{1}{3}}{12}} + \cancel{\frac{\frac{1}{3}}{9}}\right) + \left(\cancel{\frac{-\frac{1}{3}}{13}} + \cancel{\frac{\frac{1}{3}}{10}}\right) + \cdots$$

terms

$\Rightarrow S = \frac{\frac{1}{3}}{4} + \frac{\frac{1}{3}}{5} + \frac{\frac{1}{3}}{6} = \frac{1}{3}\left(\frac{1}{4} + \frac{1}{5} + \frac{1}{6}\right)$

$= \frac{1}{3}\left(\frac{15+12+10}{60}\right)$

$= \frac{1}{3}\,\frac{37}{60} = \boxed{\frac{37}{180}}$

$\downarrow$
0

This is called a telescoping sum.

Usually, we are not able to compute sums of series by hand — we need a computer to estimate. — We also need to know if a sum converges or diverges (even if we can't calculate it).

$\boxed{\text{Examples}}$

① $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots$

$\sum_{n=1}^{\infty} \frac{1}{n}$

C	D
ll	l

Notice $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots + \frac{1}{8} + \frac{1}{9} + \cdots + \frac{1}{16} + \frac{1}{17} + \cdots$

$\cdots + \frac{1}{32} + \frac{1}{33} + \cdots + \frac{1}{64} + \cdots$

$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \left(\underbrace{\frac{1}{16} + \frac{1}{16} + \cdots + \frac{1}{16}}_{8}\right) + \cdots$

$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots = \infty$

$\boxed{\text{So this diverges.}}$

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \leftarrow$ harmonic series

---

Comparison test:

If $a_k \geq b_k$ for $k = 1, 2, \dots$

then $\sum_{k=1}^{\infty} a_k \geq \sum_{k=1}^{\infty} b_k$.

If $\sum b_k = \infty$, then $\sum a_k = \infty$.

Also: If $a_k \geq b_k \geq 0$ and

$\sum_{k=1}^{\infty} a_k$ converges to S, then

$\sum_{k=1}^{\infty} b_k$ converges, and

$$S \geq \sum_{k=1}^{\infty} b_k \geq 0.$$

$\boxed{\text{Ex}}$ ① Does $\sum_{m=2}^{\infty} \frac{m + e^{-m}}{(m+1)^2}$ converge or diverge?

$$\frac{m}{(m+1)^2} \leq \frac{m + e^{-m}}{(m+1)^2}$$

$$\frac{\frac{1}{2}(m+1)}{(m+1)^2} \leq \frac{m}{(m+1)^2} \leq \frac{m+e^{-m}}{(m+1)^2}$$

$$\frac{\frac{1}{2}(m+1)}{(m+1)^2} = \frac{1}{2}\frac{1}{(m+1)}$$

and $\displaystyle\sum_{m=2}^{\infty} \frac{1}{2}\frac{1}{(m+1)}$

$$\leq \frac{1}{2}\cdot\frac{1}{3} + \frac{1}{2}\cdot\frac{1}{4} + \frac{1}{2}\cdot\frac{1}{5} + \cdots$$

$$= \frac{1}{2}\left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots\right) = \infty$$

(harmonic series)

$$\therefore \sum_{m=2}^{\infty} \frac{m+e^{-m}}{(m+1)^2} = \infty \text{ also.}$$

Ex Does this converge or diverge?

$$\sum_{p=3}^{\infty} \frac{3^p - 1}{4^p}.$$

Observe that

$$0 \leq \frac{3^p - 1}{4^p} \leq \frac{3^p}{4^p} = \left(\frac{3}{4}\right)^p.$$

So since $\displaystyle\sum_{p=3}^{\infty}\left(\frac{3}{4}\right)^p$ is a geometric series that converges

$= \left(\frac{3}{4}\right)^3 + \left(\frac{3}{4}\right)^4 + \left(\frac{3}{4}\right)^5 + \cdots$

with $r = \frac{3}{4}$, $a = \left(\frac{3}{4}\right)^3$

$$\sum_{p=3}^{\infty}\left(\frac{3}{4}\right)^p=\frac{a}{1-r}=\frac{\left(\frac{3}{4}\right)^3}{1-\frac{3}{4}}=4\cdot\left(\frac{3}{4}\right)^3$$

$\frac{1}{4}$

$$\therefore \boxed{\sum_{p=3}^{\infty}\frac{3^p-1}{4^p} \text{ converges, and}}$$

$$\boxed{0\le\sum_{p=3}^{\infty}\frac{3^p-1}{4^p}\le 4\cdot\left(\frac{3}{4}\right)^3}.$$

by the comparison test.

We can actually calculate it exactly:

$$\sum_{p=3}^{\infty}\frac{3^p-1}{4^p}=\sum_{p=3}^{\infty}\frac{3^p}{4^p}-\frac{1}{4^p}$$

$$=\sum_{p=3}^{\infty}\left(\frac{3}{4}\right)^p-\sum_{p=3}^{\infty}\left(\frac{1}{4}\right)^p$$

since all terms positive & first term converges.

geom: $a=\left(\frac{1}{4}\right)^3$, $r=\frac{1}{4}$.

$$=\frac{\left(\frac{1}{4}\right)^3}{1-\frac{1}{4}}=\frac{\left(\frac{1}{4}\right)^3}{\frac{3}{4}}$$

$$=\frac{4}{3}\left(\frac{1}{4}\right)^3.$$

Algebraic Sum theorem

$\sum a_k$ converges, $\sum b_k$ converges

$\Rightarrow \sum(a_k+b_k)=\left(\sum a_k\right)+\left(\sum b_k\right)$ converges

Thus

$$\boxed{\sum_{p=3}^{\infty}\frac{3^p-1}{4^p}=4\left(\frac{3}{4}\right)^3-\frac{4}{3}\left(\frac{1}{4}\right)^3}.$$

Quiz

1. What is your name?
2. What is your favorite movie?
3. Simplify:

   a. $\dfrac{n+1}{n^2+2n+1} - \dfrac{1}{n}$

   b. $\dfrac{n^2(2n+1)}{(2n^2-n-1)}$

   c. your life.

4. Find $\lim\limits_{n\to\infty} \dfrac{n^3+2^{-n}-\cos(3n)}{(4n)(2n^2-1)}$